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SINGULAR INTEGRALS WITH ROUGH KERNELS ALONG REAL-ANALYTIC SUBMANIFOLDS IN $\mathbf{R}^3$

DASHAN FAN, KANGHUI GUO, AND YIBIAO PAN

ABSTRACT. L^p mapping properties will be established in this paper for singular Radon transforms with rough kernels defined by translates of a real-analytic submanifold in $\mathbf{R}^3$.

1. INTRODUCTION

The main purpose of this paper is to establish the L^p boundedness of singular integral operators with rough kernels supported by real-analytic submanifolds in $\mathbf{R}^3$. For $y \in \mathbf{R}^n$, let $K(y)$ be a Calderón-Zygmund type kernel of the form

$$K(y) = b(|y|)\frac{\Omega(y)}{|y|^n}, \tag{1.1}$$

where Ω is homogeneous of degree 0, integrable over $\mathbf{S}^{n-1}$ and satisfies

$$\int_{\mathbf{S}^{n-1}} \Omega(y) d\sigma(y) = 0, \tag{1.2}$$

and $b : [0, \infty) \to \mathbf{C}$ is a measurable function. Let $d \in \mathbf{N}$ and let $\mathbf{U}$ be a neighborhood of the origin in $\mathbf{R}^n$. For a suitable mapping $\mathbf{\Psi}\colon \mathbf{U} \to \mathbf{R}^d$ we define the singular integral operator $T_{\mathbf{\Psi}}$ on $\mathbf{R}^d$ by

$$(T_{\mathbf{\Psi}} f)(x) = \text{p.v.} \int_{\mathbf{U}} f(x - \mathbf{\Psi}(y))K(y)dy. \tag{1.3}$$

When $n = d$, $\mathbf{U} = \mathbf{R}^n$ and $\mathbf{\Psi}(y) \equiv y$, $T_{\mathbf{\Psi}}$ becomes the classical singular integral operator T_I:

$$(T_I f)(x) = \text{p.v.} \int_{\mathbf{R}^n} f(x - y)K(y)dy. \tag{1.4}$$

By introducing the "method of rotations", Calderón and Zygmund proved in 1956 that the operator T_I is bounded on $L^p(\mathbf{R}^n)$ for $1 < p < \infty$ when $b \equiv 1$ and $\Omega \in L\log L(\mathbf{S}^{n-1})$. If we let $H^1(\mathbf{S}^{n-1})$ denote the Hardy space on the unit sphere, then $L\log L(\mathbf{S}^{n-1}) \subset H^1(\mathbf{S}^{n-1}) \subset L^1(\mathbf{S}^{n-1})$ and the method of Calderón and Zygmund allows the condition $\Omega \in L\log L(\mathbf{S}^{n-1})$ to be weakened to $\Omega \in H^1(\mathbf{S}^{n-1})$ for the L^p boundedness of T_I with $b \equiv 1$ (see [3], [20], [5]). On the

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other hand, Calderón and Zygmund showed that the L^p boundedness of T_I may fail if the condition $\Omega \in L\log L(\mathbf{S}^{n-1})$ is replaced by any weaker metric condition $\Omega \in L^\phi(\mathbf{S}^{n-1})$ with a ϕ satisfying $\phi(t) = o(t\log t)$ as $t \to \infty$ (e.g., $L^1 = L^\phi$ with $\phi(t) = t$).

The study of the operator T_I with a nonsmooth factor $b(|y|)$ in its kernel was initiated by R. Fefferman in [8] and continued by many authors. It is now known that the operator T_I is bounded on $L^p(\mathbf{R}^n)$ for $1 < p < \infty$ when $\Omega \in H^1(\mathbf{S}^{n-1})$ and b satisfies a very mild condition (see the condition (1.6) below).

For a general mapping $\mathbf{\Psi}$, the operator $T_{\mathbf{\Psi}}$ belongs to the class of singular Radon transforms whose L^p mapping properties are relatively well understood when the Calderón-Zygmund kernel $K(y)$ is smooth away from the origin. To understand the importance of these operators and recent advances in this area we refer the reader to Stein's survey paper [21] and his book [23], and the papers [1], [2], [13], [16]–[19].

The primary focus of our investigation is the L^p boundedness of the operator $T_{\mathbf{\Psi}}$ when $\mathbf{\Psi}$ is a general mapping and K is allowed to be nonsmooth on the unit sphere as well as in the radial direction. In the paper by the first and third authors ([9]), the L^p boundedness of $T_{\mathbf{\Psi}}$ was established for all $\mathbf{\Psi}$ defined by polynomial functions on $\mathbf{R}^n$. Below is a theorem from [9].

Theorem 1.1 ([9]). *Let $d, n \in \mathbf{N}$, $n \geq 2$, $x \in \mathbf{U} = \mathbf{R}^n$ and $\mathcal{P}(x) = (P_1(x), \dots, P_d(x))$ with $P_j(\cdot)$ being polynomials with real coefficients. Let $T_{\mathcal{P}}$ be defined by* (1.1)–(1.3). *Suppose that*

$$\Omega \in H^1(\mathbf{S}^{n-1}); \tag{1.5}$$

and

$$\sup_{R>0} \frac{1}{R}\int_0^R |b(t)|^\gamma dt < \infty, \quad \text{for some } \gamma > 1. \tag{1.6}$$

Then for $|\frac{1}{p} - \frac{1}{2}| < \min\{\frac{1}{2}, \frac{1}{\gamma'}\}$ there exists a constant $C_p > 0$ such that

$$\|T_{\mathcal{P}} f\|_{L^p(\mathbf{R}^d)} \leq C_p \|\Omega\|_{H^1(\mathbf{S}^{n-1})} \|f\|_{L^p(\mathbf{R}^d)}. \tag{1.7}$$

The constant C_p may depend on $p, d, n, b(\cdot)$ and $\deg(P_j)$, but it is independent of the coefficients of the P_j's.

The natural question is how to extend Theorem 1.1 from the submanifolds given by polynomials to the ones given by real-analytic functions in R^n. In a recent paper ([10]), we realized that if one replaces the H^1 condition (1.5) for Ω with the condition $\Omega \in L^q(\mathbf{S}^{n-1})$, $q > 1$, then the extension is relatively easy. Examining the proof of Theorem 1.1, one will see that the key idea is to reduce a polynomial of order m to a polynomial of order $m-1$, then to $m-2$ and so on. This argument obviously breaks down for real-analytic functions since in general no highest-order term(s) can be found for such functions; so new ideas are imperative if one does not wish to strengthen condition (1.5) for real-analytic submanifolds. It turns out that we need to obtain various oscillatory integral estimates which are essentially different from those developed in [9]. At the present point, we are able to get what we want only for the case $n = 2$ and $d = 3$ due to the fact that one of the oscillatory integral estimates works only for this case (see Theorem 3.7).

Also, we want to point out that even though condition (1.6) is weaker than the condition $b \in L^\infty$, the proof of Theorem 1.1 in [9] and the proof of Theorem 1.2 in [10] are only technically different for both conditions. In order to avoid some confusion of the notation, in this paper we present our following result with the condition $b \in L^\infty$ even though it holds under condition (1.6) (with a p dependent on γ as stated in (1.7)).

Theorem 1.2. *Let* $\mathbf{U}$ *be a bounded neighborhood of the origin in* $\mathbf{R}^2$ *and* $\mathbf{\Psi}$ *a real-analytic mapping from* $\mathbf{U}$ *into* $\mathbf{R}^3$ *given by* $\mathbf{\Psi}(w) = (w, \psi(w))$*. Let* $T_{\mathbf{\Psi}}$ *be defined by* (1.1)–(1.3)*. If* Ω *satisfies* (1.5) *and* $b(\cdot) \in L^\infty$*, then there exists a positive constant* $C_p = C(p, b, \mathbf{\Psi})$ *such that*

$$\|T_{\mathbf{\Psi}} f\|_{L^p(\mathbf{R}^3)} \le C_p \|\Omega\|_{H^1(\mathbf{S}^1)} \|f\|_{L^p(\mathbf{R}^3)}, \quad 1 < p < \infty. \tag{1.8}$$

The paper is organized as follows. A reduction lemma on maximal functions is given in section 2. Section 3 contains several oscillatory integral estimates including a variation of van der Corput's lemma (Lemma 3.3) which, in addition to being a key element in our proofs, may have applications elsewhere. In section 4 we use the reduction lemma obtained in section 2 to establish a key estimate of maximal functions (Theorem 4.7), from which Theorem 1.2 will be derived.

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2. A Reduction lemma on maximal functions

For a sequence of measures $\{\sigma_k\}_{k \in \mathbf{Z}}$ on $\mathbf{R}^d$ we define the operator σ^* by

$$\sigma^*(f)(x) = \sup_{k \in \mathbf{Z}} ||\sigma_k| * f(x)|.$$

For $b(\cdot) \in L^\infty$, $\Omega(\cdot) \in L^1(S^{n-1})$, let $K(w)$ be given by (1.1). For $k \in \mathbf{Z}_-$ we define $D_k \subset \mathbf{R}^n$ by

$$D_k = \{w \in \mathbf{R}^n \ | \ 2^k \le |w| < 2^{k+1}\}.$$

For a smooth mapping $\Gamma : \mathbf{R}^n \to \mathbf{R}^d$ we define the measures $\sigma_{k,\Gamma}$ on $\mathbf{R}^d$ by

$$\int_{\mathbf{R}^d} f d\sigma_{k,\Gamma} = \int_{D_k} f(\Gamma(w)) K(w) dw. \tag{2.1}$$

We also introduce positive measures $\mu_{k,\Gamma}$:

$$\int_{\mathbf{R}^d} f d\mu_{k,\Gamma} = \int_{D_k} f(\Gamma(x)) |K(w)| dw. \tag{2.2}$$

For the Fourier transforms of $\sigma_{k,\Gamma}$ and $\mu_{k,\Gamma}$, we have

$$\hat{\sigma}_{k,\Gamma}(\xi) = \int_{2^k}^{2^{k+1}} \int_{S^{n-1}} e^{i\xi\cdot\mathbf{\Gamma}(tw)} \Omega(w) d\sigma(w) \frac{b(t)}{t} dt,$$

$$\hat{\mu}_{k,\Gamma}(\xi) = \int_{2^k}^{2^{k+1}} \int_{S^{n-1}} e^{i\xi\cdot\mathbf{\Gamma}(tw)} |\Omega(w)| d\sigma(w) \frac{|b(t)|}{t} dt.$$

Also, since $\Omega \in L^1(S^{n-1})$ and $b \in \infty$, we see that $|\hat{\mu}_{k,\Gamma}(\xi)| \le C$ for all $\xi \in \mathbf{R}^d$ and $k \in \mathbf{Z}_-$. For a measurable function f on $\mathbf{R}^d$ we define $\mu^*_\Gamma(f)$ by

$$\mu^*_\Gamma(f)(y) = \sup_{k \in \mathbf{Z}_-} |(\mu_{k,\Gamma} * f)(y)|. \tag{2.3}$$

We call μ^*_Γ the maximal function associated to $\{\mu_{k,\Gamma}\}$.

Lemma 2.1. *Let $\xi \in \mathbf{R}^d,\ a_0 > 1,\ 0 < \alpha_0 < 1,\ N > 0$ and $T_0 :\ \mathbf{R}^d \to \mathbf{R}^m$ be a linear transformation and let $\Gamma_0,\ \Gamma_1 :\ \mathbf{R}^n \to \mathbf{R}^d$ be smooth mappings. If*

$$|\hat{\mu}_{k,\Gamma_0}(\xi)| \le A_0(a_0^k|T_0\xi|)^{-\alpha_0}, \tag{2.4}$$

$$|\hat{\mu}_{k,\Gamma_0}(\xi) - \hat{\mu}_{k,\Gamma_1}(\xi)| \le A_1 a_0^k |T_0\xi|, \tag{2.5}$$

$$\|\mu^*_{\Gamma_1} f\|_p \le B_p \|f\|_p, \qquad 1 < p < \infty \tag{2.6}$$

for $\xi \in \mathbf{R}^d, k \in \mathbf{Z}_-,\ |k| > N$ and $f \in L^p(\mathbf{R}^d)$, then we have

$$\|\mu^*_{\Gamma_0} f\|_p \le C_p \|f\|_p, \qquad 1 < p < \infty \tag{2.7}$$

for $f \in L^p(\mathbf{R}^d)$, where C_p may depend on $A_0,\ A_1,\ N,\ B_p,\ \alpha_0$ and a_0, but is independent of the linear transformation T_0.

Proof. Let $s = \text{rank}(T_0)$ and π_s^d be the projection operator from $\mathbf{R}^d$ to $\mathbf{R}^s$. Using Lemma 6.1 in [9], one can find a nonsingular linear transformation $G :\ \mathbf{R}^d \to \mathbf{R}^d$ such that

$$|\pi_s^d G\xi| \le |T_0\xi| \le (m - s + 1)|\pi_s^d G\xi| \tag{2.8}$$

for every $\xi \in \mathbf{R}^d$.

Let $S(\mathbf{R}^s)$ be the Schwartz class functions in $\mathbf{R}^s$. We choose and fix a function $\varphi \in C_0^\infty(\mathbf{R}^s)$ be such that $\varphi(t) \equiv 1$ for $|t| \le 1/2$, and $\varphi(t) \equiv 0$ for $|t| \ge 1$. Let $\Phi \in \mathcal{S}(\mathbf{R}^s)$ such that $\hat{\Phi} = \varphi$. Define J and $X_r = X_r(\varphi, G)$ by $(Jf)(y) = f(G^t y)$ and

$$X_r f(y) = J^{-1}((|\Phi_r| \otimes \delta_{\mathbf{R}^{d-s}}) * Jf)(y), \tag{2.9}$$

where $r > 0$ and $\Phi_r(x) = \frac{1}{r^s}\Phi(\frac{1}{r}x),\ x \in \mathbf{R}^s$. Let $X = X(\varphi, G)$ be given by

$$Xf(y) = \sup_{r>0} |X_r f(y)|. \tag{2.10}$$

It is easy to show (see Lemma 6.4 in [9]) that for $1 < p < \infty$ there exists a positive constant $C_p = C(p, \varphi, s, d)$ such that

$$\|Xf\|_p \le C_p \|f\|_p \tag{2.11}$$

for $f \in L^p(\mathbf{R}^d)$. The fact that the constant C_p is independent of the linear transformation G is crucial for the latter application of Lemma 2.1.

For $k \in \mathbf{Z}_-$ we define the measures ν_k and τ_k by

$$\hat{\nu}_k(\xi) = \varphi(a_0^k \pi_s^d G\xi)\hat{\mu}_{k,\Gamma_1}(\xi) \tag{2.12}$$

and

$$\tau_k = \mu_{k,\Gamma_0} - \nu_k. \tag{2.13}$$

We also set

$$\tau^*(f)(y) = \sup_{k \in \mathbf{Z}_-} |(|\tau_k| * f)(y)| \tag{2.14}$$

and

$$\nu^*(f)(y) = \sup_{k \in \mathbf{Z}_-} |(|\nu_k| * f)(y)|. \tag{2.15}$$

Since

$$(\nu_k * f)(y) = J^{-1}[(\Phi_{a_0^k} \otimes \delta_{\mathbf{R}^{d-s}}) * J(\mu_{k,\Gamma_1} * f)](y) \tag{2.16}$$

and

$$|\nu_k| * f(y) \leq X_{a_0^k}(\mu_{k,\Gamma_1} * |f|)(y), \tag{2.17}$$

we have

$$\nu^*(f)(y) \leq X(\mu^*_{\Gamma_1}(|f|))(y). \tag{2.18}$$

Then (2.6), (2.11) and (2.18) give

$$\|\nu^*(f)\|_p \leq C_p\|f\|_p, \quad 1 < p < \infty. \tag{2.19}$$

We now follow the *bootstrap* argument in [6] (see also [9]).

By $\varphi(0) = 1$, (2.5) and (2.8) we have

$$\begin{aligned} |\hat{\tau}_k(\xi)| &\leq |\hat{\mu}_{k,\Gamma_0}(\xi) - \hat{\mu}_{k,\Gamma_1}(\xi)| + |1 - \varphi(a_0^k \pi_s^d G\xi)||\hat{\mu}_{k,\Gamma_1}(\xi)| \\ &\leq C a_0^k |T_0\xi| \end{aligned} \tag{2.20}$$

for $k \in \mathbf{Z}_-$ and $\xi \in \mathbf{R}^d$. (2.4), (2.8), and (2.12) imply that

$$|\hat{\tau}_k(\xi)| \leq C(a_0^k|T_0\xi|)^{-\alpha_0} \tag{2.21}$$

whenever $|T_0\xi| \geq m(a_0^k)^{-1}$. Combining (2.20) and (2.21) we find

$$|\hat{\tau}_k(\xi)| \leq C[\min\{a_0^k|T_0\xi|, (a_0^k|T_0\xi|)^{-1}\}]^{\alpha_0} \tag{2.22}$$

for $k \in \mathbf{Z}_-$ and $\xi \in \mathbf{R}^d$. Set

$$g_{\Gamma_0}(f)(y) = (\sum_{k\in\mathbf{Z}_-} |\tau_k * f(y)|^2)^{\frac{1}{2}}. \tag{2.23}$$

Then by (2.13) we have

$$\mu^*_{\Gamma_0}(f)(y) \leq g_{\Gamma_0}(f)(y) + \nu^*(f)(y) \tag{2.24}$$

and

$$\begin{aligned} \tau^*(f)(y) &= \sup_{k\in\mathbf{Z}_-} [|\tau_k| * f(y)] \\ &\leq \sup_{k\in\mathbf{Z}_-} [(\mu_{k,\Gamma} * f)(y) + |\nu_k| * f(y)] \\ &\leq g_{\Gamma_0}(f)(y) + 2\nu^*(f)(y). \end{aligned} \tag{2.25}$$

Following (2.22), one can apply Plancherel's theorem to yield

$$\|g_{\Gamma_0}(f)\|_2 \leq C\|f\|_2 \tag{2.26}$$

and hence

$$\|\tau^*(f)\|_2 \leq C\|f\|_2. \tag{2.27}$$

Applying the lemma on page 544 of [6] ($q = 2,\ p_0 = 4$), we obtain

$$\|g_{\Gamma_0}(f)\|_p \leq C_p\|f\|_p \qquad \text{for } p \in (\frac{4}{3}, 4). \tag{2.28}$$

By (2.19) and (2.28) we get

$$\|\tau^*(f)\|_p \leq C_p\|f\|_p \qquad \text{for } p \in (\frac{4}{3}, 4). \tag{2.29}$$

Repeating the process used for (2.28) with $q = \frac{4}{3},\ p_0 = 8$, we get

$$\|g_{\Gamma_0}(f)\|_p \leq C_p\|f\|_p \qquad \text{for } p \in (\frac{8}{7}, 8). \tag{2.30}$$

Since this argument can continue as many times as one needs, we eventually have

$$\|g_{\Gamma_0}(f)\|_p \le C_p \|f\|_p \qquad \text{for } p \in (1,\infty). \tag{2.31}$$

Thus by (2.24) we conclude that

$$\|\mu^*_{\Gamma_0}(f)\|_p \le \|g_{\Gamma_0}(f)\|_p + \|\nu^*(f)\|_p \le C'_p \|f\|_p$$

for $1 < p < \infty$. This completes the proof of Lemma 2.1. □

Remark 2.2. What makes Lemma 2.1 useful is that given Γ_0, Γ_1, if one can verify (2.4) and (2.5), then the L^p estimate (2.7) for $\mu^*_{\Gamma_0}$ will be derived from the L^p estimate (2.6) for $\mu^*_{\Gamma_1}$. In this context we say that Γ_0 is reduced to Γ_1. Γ_1 can be further reduced to Γ_2 if one can verify (2.4) and (2.5) for Γ_1 and Γ_2 with some $a_1 > 1$, $0 < \alpha_1 < 1$ and some linear transformation T_1. This process could be extended to any finitely many steps. This is the key which enables us to prove a crucial L^p estimate of maximal functions defined by (2.3) by reducing a real-analytic mapping to a simpler mapping whose L^p estimate is previously known (see Theorem 4.7). Also, later we will classify (2.4) type estimates as *negative power estimates* and (2.5) type estimates as *positive power estimates*.

3. The oscillatory integral estimates

The following lemma is a special case of Lemma 3.2 in [15].

Lemma 3.1. *Let $\Psi \in C^\infty(\mathbf{R})$, $\phi \in C_0^\infty(a,b)$. Let $\lambda \in \mathbf{R}\setminus\{0\}$ and $l,\ m \in \mathbf{N}$ such that $|\Psi^{(l)}(x)| \le A$ for all $x \in [a,b]$ and $|\Psi^{(l+1)}(x)| \le B$ for all $x \in [a-A, b+A]$. Then there exists a constant C which depends only on l, A, B and ϕ such that*

$$\left|\int_{-\infty}^{\infty} e^{i\lambda\Psi(x)}\phi(x)dx\right| \le C|\lambda|^{-\frac{\epsilon}{l}} \int_{a-A}^{b+A} |\Psi^{(l)}(x)|^{-\epsilon(1+\frac{1}{l})}dx$$

holds for $\lambda \in \mathbf{R}$ and $\epsilon \in [0,\ 1]$.

Lemma 3.2 ([11, p. 76])**.** *Let M be a separable oriented real-analytic manifold and p a real-analytic map from M into $\mathbf{R}^n$. If S is a real-analytic subvariety of M and K is a compact subset of M, then there exists an integer d such that*

$$\operatorname{card}(K \cap S \cap p^{-1}\{y\}) \le d,$$

whenever $y \in \mathbf{R}^n$ and $\dim(S \cap p^{-1}\{y\}) \le 0$.

Lemma 3.3. *For $t \in \mathbf{R}$, $x \in \mathbf{R}^m$, let $f(t,x)$ be real-analytic for $1 \le t \le 2$ and $|x| \le \delta$. For $L \in \mathbf{N}$, if $\sum_{j=1}^{L} \left|\frac{\partial^j f}{\partial t^j}(t,x)\right| \ge A(x) \ge 0$ for all $1 \le t \le 2$ and $|x| \le \delta$, then*

$$\left|\int_1^2 e^{if(t,x)}dt\right| \le C(A(x))^{-\frac{1}{L}},$$

where C is independent of x.

Proof. For a given x, if $A(x) = 0$, the lemma holds trivially; so assume $A(x) > 0$. For positive integers j,l with $1 \le j < l \le L$, let $F_{jl}(t,x) = \frac{\partial^j}{\partial t^j}f(t,x) - \frac{\partial^l}{\partial t^l}f(t,x)$. Let $\Sigma_{jl} = \{x \mid F_{jl}(t,x) \ne 0 \text{ for some } t \in [1,2]\}$. Applying Lemma 3.2 with

$M = \mathbf{R}^{m+1},\ S = \{(t,x) \mid F_{jl}(t,x) = 0\},\ p:\ (t,x) \to x$, we see that there exists $d_{jl} \in \mathbf{N}$ such that

$$\max_{x\in\Sigma_{jl}} \operatorname{card}(\{t \mid F_{jl,x}(t) = 0,\ \ t \in [1,2]\}) \le d_{jl}.$$

Let $D = \sum_{j,l=1}^{L} d_{jl}$. Then a standard argument yields that for all x with $A(x) > 0$, one can decompose the interval $[1,2]$ into at most D subintervals such that on each subinterval, $|\frac{\partial^j}{\partial t^j} f(t,x)| \ge \frac{1}{L} A(x)$ for some $j,\ 1 \le j \le L$. The conclusion of the lemma follows from van der Corput's lemma (except for $j = 1$, when one uses integration by parts directly) and the trivial estimate $|\int_a^b e^{if(t,x)}dt| \le 1$ for any $1 \le a < b \le 2$.

This finishes the proof of the lemma. □

We need some notation. Let $B_m^1 = \{y \in \mathbf{R}^m;\ \ |y| < 1\}$. Let $t \in \mathbf{R},\ x = (x_1, x'') \in \mathbf{R} \times \mathbf{R}^{n-2},\ w = (x, x_n) \in \mathbf{R}^{n-1} \times \mathbf{R}$. Let M_n be the set of all rotations in $\mathbf{R}^n$. It is well-known that M_n can be embedded as a smooth compact submanifold in $\mathbf{R}^{n^2}$. Let $\mathbf{Z}_-$ be the set of negative integers.

Let $\mathbf{e} = (0, 0, \cdots, 1)$ be the north pole of $\mathbf{S}^{n-1},\ \Omega(\cdot) \in L^1(\mathbf{S}^{n-1})$ and $b \in L^\infty$. Let $\mathbf{F}:\ \mathbf{R}^n \to \mathbf{R}^s$ be real-analytic on B_n^1. For $\eta \in \mathbf{R}^s \setminus \{0\}$ and $\tilde{r} \in M_n$, we define $\mathbf{F}_{\tilde{r}}$ by $\mathbf{F}_{\tilde{r}}(w) = \mathbf{F}(\tilde{r}w)$ for any $w \in R^n$. Furthermore, define a scalar-valued function $F_{\tilde{r},\eta'}$ by $F_{\tilde{r},\eta'}(w) = \eta' \cdot \mathbf{F}(\tilde{r}w)$ for any $w \in R^n$, where $\eta' = \frac{\eta}{|\eta|}$.

For $k \in \mathbf{Z}_-$, set

$$P_{k,F_{\tilde{r}}}(\eta) = \int_{2^k}^{2^{k+1}} \int_{\mathbf{S}^{n-1}} e^{i|\eta| F_{\tilde{r},\eta'}(tw)} \Omega(w) d\sigma(w) \frac{b(t)}{t} dt, \tag{3.1}$$

$$Q_{k,F_{\tilde{r}}}(\eta) = \int_1^2 |\int_{\mathbf{S}^{n-1}} e^{i|\eta| F_{\tilde{r},\eta'}(2^k tw)} \Omega(w) d\sigma(w)|^2 dt. \tag{3.2}$$

In (3.1) one can dilate the variable t and apply Hölder's inequality to control $P_{k,F_{\tilde{r}}}(\eta)$ in terms of $Q_{k,F_{\tilde{r}}}(\eta)$.

Lemma 3.4. *Let $P_{k,F_{\tilde{r}}}(\eta)$ and $Q_{k,F_{\tilde{r}}}(\eta)$ be as in* (3.1) *and* (3.2). *Then*

$$|P_{k,F_{\tilde{r}}}(\eta)| \le C[Q_{k,F_{\tilde{r}}}(\eta)]^{\frac{1}{2}}.$$

Lemma 3.5. *Let $\Omega \in L^\infty(\mathbf{S}^{n-1})$ with $\operatorname{supp}(\Omega) \subset \{w \in S^{n-1};\ |w - \mathbf{e}| < \rho\}$ and $\|\Omega\|_\infty \le \rho^{-n+1}$. For $\eta \in R^s \setminus \{0\}$ and $\tilde{r} \in M_n$, let*

$$g(t, x, \eta', \tilde{r}) = F_{\tilde{r},\eta'}(tx,\ t\sqrt{1 - |x|^2}).$$

Assume that for $\eta_0 \in R^s \setminus \{0\}$ and $\tilde{r}_0 \in M_n$, there exist $l \ge 1,\ m \ge 1$ and $A_0 \ne 0$ such that

$$\frac{\partial^j}{\partial x_1^j} \frac{\partial^l g}{\partial t^l}(0, 0, \eta_0', \tilde{r}_0) = 0, \qquad \text{for} \quad 0 \le j \le m-1; \tag{3.3}$$

$$\frac{\partial^m}{\partial x_1^m} \frac{\partial^l g}{\partial t^l}(0, 0, \eta_0', \tilde{r}_0) = A_0. \tag{3.4}$$

Then one can find $N_0 > 0,\ 0 < R_0 \le \frac{1}{2},\ U_{\eta_0'},$ and $V_{\tilde{r}_0}$ such that

$$Q_{k,F_{\tilde{r}}}(\eta) \le C_0(|\eta| 2^{kl(l+1)} \rho^{m(l+1)})^{-\frac{1}{4lm}}$$

if $k \in \mathbf{Z}_-$, $|k| > N_0$, $0 < \rho \le R_0$, $\eta' \in U_{\eta_0'}$, and $\tilde{r} \in V_{\tilde{r}_0}$, where $U_{\eta_0'}$ and $V_{\tilde{r}_0}$ are small neighborhoods of η_0' and $\tilde{r}_0$ in $\mathbf{S}^{s-1}$ and M_n respectively. Also, the constants C_0, N_0, R_0 may depend on $\eta_0', \tilde{r}_0$ (and hence on l, m, and A_0), but are independent of k and ρ.

Proof. Let $f(x) = \Omega(x, \sqrt{1-|x|^2})(1-|x|^2)^{-\frac{1}{2}}$ and choose $\phi \in C_0^\infty(0,\ 3)$ such that $\phi(t) \ge 0$ for all $t \in \mathbf{R}$ and $\phi(t) = 1$ for $1 \le t \le 2$. We have

$$\begin{aligned} Q_{k,F_{\tilde{r}}}(\eta) &\le \int_1^2 |\int_{B_\rho^{n-1}} e^{i|\eta| g(2^k t, x, \eta', \tilde{r})} f(x) dx|^2 \phi(t) dt \\ &\le \int_{B_\rho^{n-1}} \int_{B_\rho^{n-1}} [\int_{-\infty}^{\infty} e^{i|\eta|[g(2^k t, x, \eta', \tilde{r}) - g(2^k t, y, \eta', \tilde{r})]} \phi(t) dt] f(x) \overline{f(y)} dx dy. \end{aligned}$$

Denote $I_k(t, x_1, x'', y, \eta', \tilde{r}) = 2^{-kl}[g(2^k t, x, \eta', \tilde{r}) - g(2^k t, y, \eta', \tilde{r})]$. Then (3.3) and (3.4) give

$$\begin{aligned} &\frac{\partial^j}{\partial x_1^j} \frac{\partial^l I_k}{\partial t^l}(0,0,0,0,\eta',\tilde{r}) = 0, \qquad \text{for} \quad 0 \le j \le m-1, \\ &\frac{\partial^m}{\partial x_1^m} \frac{\partial^l I_k}{\partial t^l}(0,0,0,0,\eta',\tilde{r}) = A_0. \end{aligned}$$

Invoking the Malgrange Preparation Theorem ([12]), we see that there exist $N_0 > 0$, $0 < R_0 \le \frac{1}{2}$, $U_{\eta_0'}$, and $V_{\tilde{r}_0}$ such that

$$\frac{\partial^l I_k}{\partial t^l}(t, x_1, x'', y, \eta', \tilde{r}) = C(2^k t, x'', y, \eta', \tilde{r})[x_1^m + \sum_{j=1}^{m-1} b_j(2^k t, x'', y, \eta', \tilde{r}) x_1^j]$$

for $k \in \mathbf{Z}_-$, $|k| > N_0$, $|x_1| \le \rho \le R_0$, $|x''| \le \rho \le R_0$, $|y| \le \rho \le R_0$, $\eta' \in U_{\eta_0'}$, and $\tilde{r} \in V_{\tilde{r}_0}$, where $|C(2^k t, x'', y, \eta', \tilde{r})| \ge \frac{1}{2}|A_0|$. □

Applying Lemma 3.1 for $\epsilon = \frac{1}{4m}$, we have

$$\begin{aligned} &Q_{k,F_{\tilde{r}}}(\eta) \\ &\quad \le C(2^{kl}|\eta|)^{-\frac{1}{4lm}} \int_{B_\rho^{n-1}} \int_{B_\rho^{n-1}} \int_{-A}^{3+A} [|x_1^m + \sum_{j=1}^{m-1} b_j(2^k t, x'', y, \eta', \tilde{r}) x_1^j|]^{-\frac{l+1}{4lm}} dt \\ &\qquad \cdot |f(x) f(y)| dx dy \\ &\quad \le C(2^{kl}|\eta|)^{-\frac{1}{4lm}} \int_{-A}^{3+A} \int_{B_\rho^{n-1}} \int_{|x''| \le \rho} \int_{|x_1| \le \rho} \\ &\qquad |x_1^m + \sum_{j=1}^{m-1} b_j(2^k t, x'', y, \eta', \tilde{r}) x_1^j|^{-\frac{l+1}{4lm}} \cdot |f(x_1, x'') f(y)| dx_1 dx'' dy dt \\ &\quad \le C(2^{kl}|\eta|)^{-\frac{1}{4lm}} \int_{-A}^{3+A} \int_{B_\rho^{n-1}} \int_{|x''| \le \rho} \int_{|x_1| \le 1} \\ &\qquad |(\rho x_1)^m + \sum_{j=1}^{m-1} b_j(2^k t, x'', y, \eta', \tilde{r})(\rho x_1)^j|^{-\frac{l+1}{4lm}} \cdot \rho^{-n+1} f(y)| \rho dx_1 dx'' dy dt \\ &\quad \le C(2^{kl}|\eta| \rho^{m(l+1)})^{-\frac{1}{4lm}}, \end{aligned}$$

where C is independent of k, ρ.

Here we used the assumption $\|\Omega\|_\infty \le \rho^{-n+1}$ and the facts that $\frac{l+1}{4l} < 1$ and the constants A and B in Lemma 2.2 could be chosen to be independent of k and ρ since $k \in \mathbf{Z}_-$ and $\rho \le \frac{1}{2}$. This ends the proof of Lemma 3.5.

Theorem 3.6. *Let $g(t,x,\eta',\tilde{r})$ be defined as in Lemma 3.5. If for each $\eta' \in \mathbf{S}^{n-1}$, $\tilde{r} \in M_n$, there exist $l,\ m,\ A$ (may depend on η' and $\tilde{r}$) such that (3.3) and (3.4) hold, then one can find $L, M \ge 1,\ N > 0,\ 0 < R \le \frac{1}{2}$ such that*

$$|P_{k,F_{\tilde{r}}}(\eta)| \le C(|\eta| 2^{kL}\rho^{M(L+1)})^{-\frac{1}{8LM}} \tag{3.5}$$

if $k \in \mathbf{Z}_-,\ |k| > N,\ \ 0 < \rho \le R$, where C is independent of $k,\ \rho,\ \eta$ and $\tilde{r}$.

Proof. From Lemma 3.4, it suffices to show

$$Q_{k,F_{\tilde{r}}}(\eta) \le C(|\eta| 2^{kL}\rho^{M(L+1)})^{-\frac{1}{4LM}}. \tag{3.6}$$

Let $i,\ j \in \mathbf{N}$. For any $\eta_i \in \mathbf{R}^s \setminus \{0\}$ and $\tilde{r}_j \in M_n$, Lemma 3.5 asserts that there exist $l_{i,j} \ge 1,\ m_{i,j} \ge 1,\ N_{i,j} > 0,\ 0 < R_{i,j} \le \frac{1}{2},\ \ U_{(\eta_i)'}$ and $V_{\tilde{r}_j}$ such that

$$Q_{k,F_{\tilde{r}}}(\eta) \le C_{i,j}(|\eta| 2^{kl_{i,j}}\rho^{m_{i,j}(l_{i,j}+1)})^{-\frac{1}{4l_{i,j}m_{i,j}}}$$

if $k \in \mathbf{Z}_-,\ |k| > N_{i,j},\ \ 0 < \rho \le R_{i,j},\ \ (\eta)' \in U_{(\eta_i)'}$ and $\tilde{r} \in V_{\tilde{r}_j}$.

One then finds $m \ge 1$ such that $\bigcup_{i=1}^m U_{(\eta_i)'} = \mathbf{S}^{s-1}$ and $\bigcup_{j=1}^m V_{\tilde{r}_j} = M_n$. Using the trivial estimate $Q_{k,F_{\tilde{r}}}(\eta) \le C$, we obtain (3.6) by taking $L = \max_{1\le i,\ j \le m}\{l_{i,j}\}$, $M = \max_{1\le i,\ j\le m}\ \{m_{i,j}\},\ N = \max_{1\le i,\ j\le m}\ \{N_{i,j}\},\ R = \min_{1\le i,\ j\le m}\ \{R_{i,j}\}$, $C = \max_{1\le i,\ j\le m}\ \{C_{i,j}\}$. □

The proof of Theorem 3.6 is complete.

For real x and t, let $f(t,x) = \sum_{j=1}^\infty t^j[\sum_{n=0}^\infty a_{j,n}x^n]$, where $a_{j,n}$ are real numbers satisfying

$$\sum_{j=2}^{\infty}(\frac{1}{2})^j[\sum_{n=0}^{\infty}|a_{j,n}|(\frac{1}{2})^n] < \infty. \tag{3.7}$$

For $|y| \le \frac{1}{4},\ |x| \le \frac{1}{4}$, we rewrite $f(t,x)$ as $\sum_{j=1}^\infty t^j[\sum_{m=0}^\infty b_{j,m}(y)(x-y)^m]$, where

$$b_{j,m}(y) = \frac{1}{m!}\sum_{n=m}^{\infty} n(n-1)\cdots(n-m+1)a_{j,n}y^{n-m}. \tag{3.8}$$

Since $\frac{d^l}{dy^l}(b_{j,m}(y)) = (m+l)(m+l-1)\cdots(m+1)b_{j,m+l}(y)$ and

$$\begin{aligned}
&\sum_{m=0}^{\infty}\frac{1}{m!}[\sum_{n=m}^{\infty} n(n-1)\cdots(n-m+1)|a_{j,n}|(\frac{1}{4})^{n-m}](\frac{1}{4})^m \\
= &\sum_{n=0}^{\infty}[\sum_{m=0}^{n}\frac{1}{m!}n(n-1)\cdots(n-m+1)(\frac{1}{2})^n]|a_{j,n}|(\frac{1}{2})^n \\
= &\sum_{n=0}^{\infty}|a_{j,n}|(\frac{1}{2})^n,
\end{aligned}$$

it follows from (3.7) and (3.8) that if $|y| \le \frac{1}{16},\ |x| \le \frac{1}{16}$, then for each nonnegative integer l there exists a constant C_l independent of x and y such that

$$\sum_{j=2}^{\infty}(\frac{1}{2})^j[\sum_{m=0}^{\infty}|\frac{d^l}{dy^l}(b_{j,m}(y))||x-y|^m] \le C_l. \tag{3.9}$$

For $\bar{L} \geq 2,\ \bar{M} \geq 1$, define

$$\psi(t,x,y) = \sum_{j=1}^{\bar{L}-1} t^j [\sum_{m=0}^{\infty} b_{j,m}(y)(x-y)^m] + \sum_{j=\bar{L}}^{\infty} t^j [\sum_{m=0}^{\bar{M}} b_{j,m}(y)(x-y)^m].$$

Let $\eta(y)$ be a continuous function of y. For $k \in \mathbf{Z}_-,\ \alpha(\cdot) \in L^1(\mathbf{R})$ and $b(\cdot) \in L^\infty$, set

$$I_k(y) = \int_{2^k}^{2^{k+1}} \int_{-\frac{1}{16}}^{\frac{1}{16}} e^{i\eta(y)\psi(t,x,y)} \alpha(x) dx \frac{b(t)}{t} dt, \tag{3.10}$$

$$Q_k(y) = \int_1^2 | \int_{-\frac{1}{16}}^{\frac{1}{16}} e^{i\eta(y)\psi(2^k t,x,y)} \alpha(x) dx|^2 dt. \tag{3.11}$$

Theorem 3.7. *Given $y \in [-\frac{1}{16}, \frac{1}{16}]$ and $0 < \rho \leq \frac{1}{16}$, let $\alpha(\cdot) \in L^\infty(\mathbf{R})$ be such that $\mathrm{supp}(\alpha) \subset \{x \in \mathbf{R};\ |x-y| < \rho\}$ and $\|\alpha\|_\infty \leq \rho^{-1}$. If $\bar{L} > \bar{M}$ and*

$$\sum_{j=\bar{L}}^{\infty} \sum_{m=0}^{\bar{M}} |b_{j,m}(y)| > 0$$

for each $y \in [-\frac{1}{16},\ \frac{1}{16}]$, then one can find $L' \geq \bar{L},\ N > 0$ such that

$$|I_k(y)| \leq C(|\eta(y)| 2^{kL'} \rho^{\bar{M}})^{-\frac{1}{2L'}} \tag{3.12}$$

if $k \in \mathbf{Z}_-,\ |k| > N$, where C is independent of $k,\ \eta,\ y$ and ρ.

Proof. As in the proof of Theorem 3.6, (3.12) will be proved if one can prove

$$|Q_k(y)| \leq C(|\eta(y)| 2^{kL'} \rho^{\bar{M}})^{-\frac{1}{L'}}. \tag{3.13}$$

We note that since both $\eta(y)$ and $\mathbf{Q}_k(y)$ are continuous, one only needs to prove (3.13) with finitely many points removed from the interval $[-\frac{1}{16},\ \frac{1}{16}]$, as long as this finite number is independent of $k,\ \eta,$ and ρ. For $j \geq \bar{L}$, let $\mathbf{v}_j(y) = (b_{j,0}(y),\ b_{j,1}(y), \cdots, b_{j,\bar{M}}(y))$ and $S(y)$ be the linear space spanned by all $\mathbf{v}_j(y),\ j \geq \bar{L}$. For each $y \in [-\frac{1}{16},\ \frac{1}{16}]$, let $d(y)$ be the dimension of $S(y)$ and $d = \max_{|y| \leq \frac{1}{16}} d(y)$. Since $1 \leq d(y) \leq \bar{M}+1$ for each y, one sees that $1 \leq d \leq \bar{M}+1$. We pick a $y_0 \in [-\frac{1}{16},\ \frac{1}{16}]$ with $d = d(y_0)$. For any $\{j_1,\ j_2,\ \cdots, j_d\}$ with $\bar{L} \leq j_1 < j_2 < \cdots < j_d < \infty$, let $J = \{j_1,\ j_2, \cdots, j_d\}$. For any $y \in [-\frac{1}{16},\ \frac{1}{16}]$, we form an $(\bar{M}+1) \times d$ matrix $M_J(y)$ by choosing the i-th column of $M_J(y)$ to be $\mathbf{v}_{j_i}(y)$. If $\{\mathbf{v}_{j_1}(y),\ \mathbf{v}_{j_2}(y), \cdots, \mathbf{v}_{j_d}(y)\}$ is a basis for $S(y)$, then one can find an invertible $d(y) \times d(y)$ submatrix of $M_J(y)$. To simplify the notation, we assume that $\{\mathbf{v}_{\bar{L}}(y_0),\ \mathbf{v}_{\bar{L}+1}(y_0), \cdots, \mathbf{v}_{\bar{L}+d-1}(y_0)\}$ is a basis of $S(y_0)$. We write $J_0 = \{\bar{L},\ \bar{L}+1, \cdots, \bar{L}+d-1\}$. We may also assume that the first d rows of $M_{J_0}(y_0)$ give an invertible $d \times d$ submatrix at y_0. For a given y and J, let $A_J(y)$ be the submatrix of $M_J(y)$ whose entries are the first d rows of $M_J(y)$. Since $det(A_{J_0}(y))$ is real-analytic for $|y| \leq \frac{1}{16}$ and is not identically zero since $det(A_{J_0}(y_0)) \neq 0$, we see that $det(A_{J_0}(\cdot))$ has only finitely many zeros in $[\frac{1}{16}, \frac{1}{16}]$. We cut $[\frac{1}{16}, \frac{1}{16}]$ into several subintervals, each of which contains only one zero of $det(A_{J_0}(\cdot))$. Without loss of generality, we may assume that the only possible zero of $det(A_{J_0}(y))$ is $y = 0$. For

$j > \bar{L} + d - 1$ and $y \neq 0$, let $A^{j,i}_{J_0}(y)$ be the matrix obtained by replacing the i-th column of $A_{J_0}(y)$ by the first d components of $\mathbf{v}_j(y)$. We are able to write

$$\mathbf{v}_j(y) = \sum_{i=1}^{d} k_{j,i}(y)\mathbf{v}_{\bar{L}+i-1}(y), \tag{3.14}$$

where $k_{j,i}(y) = \frac{det(A^{j,i}_{J_0}(y))}{det(A_{J_0}(y))}$.

For each J, we write

$$det(A_J(y)) = \sum_{u=0}^{\infty} \beta_{J,u} y^u.$$

Based on the property of $det(A_{J_0}(y))$ one can find some $\bar{J}$ and some nonnegative integer s such that $\beta_{\bar{J},s} \neq 0$, but $\beta_{J,u} = 0$, for all J and $0 \leq u \leq s-1$. Let the column vectors of $M_{\bar{J}}(y)$ be $\{\mathbf{v}_{\bar{j}_1}(y),\ \mathbf{v}_{\bar{j}_2}(y), \cdots, \mathbf{v}_{\bar{j}_d}(y)\}$ with $\bar{j}_d \geq \bar{L} + d - 1$.

From the choice of $\bar{J}$ and (3.10), we see that there exists $C > 0$, $\epsilon > 0$ such that for all $|y| \leq \epsilon$ and all $j > \bar{j}_d$, we have

$$|det(A_{\bar{J}}(y))| \geq C|y|^s, \tag{3.15}$$

$$(\frac{1}{2})^j |det(A^{j,i}_{\bar{J}}(y))| \leq C|y|^s, \tag{3.16}$$

where C is independent of y and j.

Hence for $j > \bar{j}_d$ and $|y| \leq \epsilon$, we have

$$\mathbf{v}_j(y) = \sum_{i=1}^{d} k_{j,i}(y)\mathbf{v}_{\bar{j}_i}(y), \tag{3.17}$$

where $k_{j,i}(y) = \frac{det(A^{j,i}_{\bar{J}}(y))}{det(A_{\bar{J}}(y))}$.

It follows from (3.15), (3.16) and (3.17) that for $j > \bar{j}_d,\ u,\ v \in [-\frac{1}{16}, \frac{1}{16}]$ and $|y| \leq \epsilon$,

(3.18)
$$(\frac{1}{2})^j |\sum_{m=0}^{\bar{M}} b_{j,m}(y)(u^m - v^m)| \leq C \sum_{i=1}^{d} |\sum_{m=0}^{\bar{M}} b_{\bar{j}_i,m}(y)(u^m - v^m)|,$$

where C is independent of $u,\ v,\ y$ and j.

For $u,\ v \in [-\frac{1}{16}, \frac{1}{16}]$ and $|y| \geq \epsilon$, (3.9) and (3.14) imply that for all $j > \bar{L}+d-1$,

(3.19)
$$(\frac{1}{2})^j |\sum_{m=0}^{\bar{M}} b_{j,m}(y)(u^m - v^m)| \leq C \sum_{l=\bar{L}}^{\bar{L}+d-1} |\sum_{m=0}^{\bar{M}} b_{l,m}(y)(u^m - v^m)|,$$

where C is independent of $u,\ v,\ y$ and j.

Combining (3.18) and (3.19) together, we see that for $u,\ v \in [-\frac{1}{16}, \frac{1}{16}],\ |y| \leq \frac{1}{16}$ and all $j > \bar{j}_d$,

(3.20)
$$(\frac{1}{2})^j |\sum_{m=0}^{\bar{M}} b_{j,m}(y)(u^m - v^m)| \leq C \sum_{l=1}^{\bar{j}_d} |\sum_{m=0}^{\bar{M}} b_{l,m}(y)(u^m - v^m)|$$

holds, where C is independent of $u,\ v,\ \ y$, and j.

Let $\alpha_y(u) = \alpha(u+y)$ and

$$\psi_y(t,u) = \sum_{j=1}^{\bar{L}-1} t^j [\sum_{m=0}^{\infty} b_{j,m}(y)u^m] + \sum_{j=\bar{L}}^{\infty} t^j [\sum_{m=0}^{\bar{M}} b_{j,m}(y)u^m].$$

One implication of (3.20) is that there exist $C > 0,\ N > 0$ such that for $k \in \mathbf{Z}_-,\ |k| > N,\ t \in [1,2],\ u,\ v \in [-\frac{1}{16}, \frac{1}{16}]$ and $|y| \le \frac{1}{16}$, we have

$$\begin{aligned} &\sum_{l=\bar{L}}^{\bar{j}_d} |\frac{\partial^{(l)}}{\partial t^{(l)}} \psi_y(2^k t, u) - \psi_y(2^k t, v)| \\ &\qquad \ge C 2^{-k\bar{j}_d} \sum_{l=\bar{L}}^{\bar{j}_d} |\sum_{m=0}^{\bar{M}} b_{l,m}(y)(u^m - v^m)|, \end{aligned} \tag{3.21}$$

where C is independent of $k,\ t,\ u,\ v$ and y.

We rewrite $Q_k(y)$ as

$$Q_k(y) = \int_{-\rho}^{\rho}\int_{-\rho}^{\rho}\int_1^2 e^{i\eta(y)[\psi_y(2^k t,u) - \psi(2^k t,v)]} dt\ \alpha_y(u)\overline{\alpha_y(v)} du dv. \tag{3.22}$$

Then (3.21), the assumptions on $\alpha(\cdot)$, Lemma 3.3 and a result of Ricci and Stein in [18] yield

$$\begin{aligned} |Q_k(y)| &\le C(|\eta(y)|2^{k\bar{j}_d})^{-\frac{1}{\bar{j}_d}} \int_{-\rho}^{\rho}\int_{-\rho}^{\rho} [\sum_{l=\bar{L}}^{\bar{j}_d} |\sum_{m=0}^{\bar{M}} b_{l,m}(y)u^m \\ &\quad - \sum_{m=0}^{\bar{M}} b_{l,m}(y)v^m|]^{-\frac{1}{\bar{j}_d}} \rho^{-2} du dv \\ &\le C(|\eta(y)|2^{k\bar{j}_d})^{-\frac{1}{\bar{j}_d}} \int_{-1}^{1}[\int_{-1}^{1} [\sum_{l=\bar{L}}^{\bar{j}_d} |\sum_{m=0}^{\bar{M}} b_{l,m}(y)\rho^m u^m \\ &\quad - \sum_{m=0}^{\bar{M}} b_{l,m}(y)\rho^m v^m|]^{-\frac{1}{\bar{j}_d}} du] dv \\ &\le C(|\eta(y)|2^{k\bar{j}_d})^{-\frac{1}{\bar{j}_d}} [\sum_{l=\bar{L}}^{\bar{j}_d} \sum_{m=0}^{\bar{M}} |b_{l,m}(y)\rho^m|]^{-\frac{1}{\bar{j}_d}} \\ &\le C(|\eta(y)|2^{k\bar{j}_d}\rho^{\bar{M}})^{-\frac{1}{\bar{j}_d}} [\sum_{l=\bar{L}}^{\bar{j}_d} \sum_{m=0}^{\bar{M}} |b_{l,m}(y)|]^{-\frac{1}{\bar{j}_d}} \\ &\le C(|\eta(y)|2^{k\bar{j}_d}\rho^{\bar{M}})^{-\frac{1}{\bar{j}_d}}, \end{aligned}$$

where C is independent of $k,\ \eta,\ y$ and ρ.

Here in the third inequality, we used the assumption that $\bar{L} > \bar{M}$ and in the last inequality, we used (3.21) and the assumption that $\sum_{j=\bar{L}}^{\infty} \sum_{m=0}^{\bar{M}} |b_{j,m}(y)| > 0$ for each $y \in [-\frac{1}{16},\ \frac{1}{16}]$.

The proof of Theorem 3.7 will be finished if one chooses L' to be $\bar{j}_d$. □

4. The proof of main results

Definition 4.1. A function $a(\cdot)$ on $\mathbf{S}^{n-1}$ is called a regular atom if there exist $\zeta \in \mathbf{S}^{n-1}$ and $\rho \in (0,2]$ such that

$$\mathrm{supp}(a) \subset \{w \in \mathbf{S}^{n-1}: \ |w-\zeta| < \rho\};$$
$$\|a\|_\infty \le \rho^{-1};$$
$$\int_{\mathbf{S}^{n-1}} a(w)d\sigma(w) = 0.$$

Let $\mathcal{S}(\mathbf{S}^{n-1})$ be the Schwartz space of smooth functions on $\mathbf{S}^{n-1}$ and $\mathcal{S}'(\mathbf{S}^{n-1})$ its dual. For $f \in \mathcal{S}'$ and the Poisson kernel $P_{ry}(x) = \frac{1-r^2}{|ry-x|^n}$ on $\mathbf{S}^{n-1}$, we define its radial maximal function P^+f by

$$P^+f(x) = \sup_{0\le r<1} |\int_{\mathbf{S}^{n-1}} P_{rx}(y)f(y)d\sigma(y)|$$

where $x \in \mathbf{S}^{n-1}$. The Hardy space $H^1(\mathbf{S}^{n-1})$ is defined by

$$H^1(\mathbf{S}^{n-1}) = \{f \in \mathcal{S}'(\mathbf{S}^{n-1}): \ \ \|P^+f\|_{L^1(\mathbf{S}^{n-1})} < \infty\}$$

with $\|f\|_{H^1(\mathbf{S}^{n-1})} = \|P^+f\|_{L^1(\mathbf{S}^{n-1})}$. We will make use of the following property of $H^1(\mathbf{S}^{n-1})$ (see [3] or [4]).

Lemma 4.2. *For any $f \in H^1(\mathbf{S}^{n-1})$ with $\int_{S^{n-1}} f(w)d\sigma(w) = 0$, there are complex numbers c_j and regular atoms a_j such that*

$$f = \sum_j c_j a_j$$

and $\|f\|_{H^1} \approx \sum_j |c_j|$.

Using (2.8) in section 2, one can easily verify the following extension of a result due to Duoandikoetxea and Rubio de Francia [6].

Lemma 4.3. *Let d and m be two positive integers and $L : \mathbf{R}^d \to \mathbf{R}^m$ a linear transformation. Suppose $\{\sigma_k\}_{k\in\mathbf{Z}}$ is a sequence of measures on $\mathbf{R}^d$ satisfying*

(i) $\|\sigma_k\| \le 1$ *for $k \in \mathbf{Z}$.*

(ii) $|\hat{\sigma}_k(\xi)| \le C[\min\{a_{kxc}|L\xi|, (a_k|L\xi|)^{-1}\}]^\alpha$ *for $\xi \in \mathbf{R}^d$ and $k \in \mathbf{Z}$.*

(iii) *For some $1 < q < \infty$, the operator $\sigma^* : f \to \sigma^*(f)$ is bounded from $L^q(\mathbf{R}^d)$ to itself.*

Then for $p \in (\frac{2q}{q+1}, \frac{2q}{q-1})$, there exists a constant $C_p = C(p,d,m)$ which is independent of L such that

$$\|\sum_{k=-\infty}^{\infty} \sigma_k * f\|_p \le C_p\|f\|_p$$

and

$$\|(\sum_{k=-\infty}^{\infty} |\sigma_k * f|^2)^{\frac{1}{2}}\|_p \le C_p\|f\|_p$$

for every $f \in L^p(\mathbf{R}^d)$.

Let $\Gamma(t): \ [0,1] \to \mathbf{R}$. For $f \in L^p(\mathbf{R}^2)$ for some $1 \le p \le \infty$, we define

$$(M_\Gamma f)(x_1,x_2) = \sup_{0<h<1} \frac{1}{h}\int_{0<t<h} |f(x_1-t, x_2-\Gamma(t))|dt.$$

Lemma 4.4 ([24, Theorem 12 (B)]). *If Γ is analytic, then*

$$\|M_\Gamma f\|_p \le C_p \|f\|_p, \quad 1 < p < \infty.$$

For a positive integer m we let V_m denote the space of real-valued homogeneous polynomials of degree m on $\mathbf{R}^n$ and set $N_m = \dim(V_m)$. For

$$P(y) = \sum_{|\alpha|=m} a_\alpha y^\alpha \in V_m$$

we define

$$\|P\| = \sum_{|\alpha|=m} |a_\alpha|.$$

If m is an even, positive integer, then we have

$$|x|^m = (x_1^2 + x_2^2 + \cdots + x_n^2)^{m/2} \in V_m.$$

We now choose a basis $\{\zeta_1, \zeta_2, \ldots, \zeta_{N_m}\}$ for the space V_m such that $\zeta_1(x) = |x|^m$ for $x \in \mathbf{R}^n$. Clearly, there are constants K_1, K_2 such that

$$K_1 \sum_{j=1}^{N_m} |c_j| \le \|P\| \le K_2 \sum_{j=1}^{N_m} |c_j|$$

for every

$$P = \sum_{j=1}^{N_m} c_j \zeta_j \in V_m.$$

For the above polynomial P, we define the linear transformation $Y_m : V_m \to V_m$ by

$$Y_m(P) = \sum_{j=2}^{N_m} c_j \zeta_j.$$

Also define the linear transformation $Z_m : V_m \to V_m$ by

$$Z_m = \begin{cases} id_{V_m} & \text{if } m \text{ is odd,} \\ Y_m & \text{if } m \text{ is even.} \end{cases}$$

Lemma 4.5 ([9, Proposition 5.1]). *Let $b \in L^\infty$, $\Omega \in L^2(\mathbf{S}^{n-1})$. Suppose $F : \mathbf{R}^n \to \mathbf{R}$ is a function given by*

$$\sum_{j=0}^{l} P_j(w) + W(|w|), \tag{4.1}$$

where $P_j(\cdot)$ is a homogeneous polynomial of degree j, $0 \le j \le l$, and $W(\cdot)$ is an arbitrary function. Then we have

$$\begin{aligned} \int_{2^k}^{2^{k+1}} \Big| \int_{\mathbf{S}^{n-1}} e^{iF(tw)} \Omega(w) d\sigma(w) \Big| \frac{|b(t)|}{t} dt \\ \le C (2^{kl} \|Z_l(P_l)\|)^{-\frac{1}{8l}} \|\Omega\|_2 \end{aligned} \tag{4.2}$$

for all $k \in \mathbf{Z}$. The constant C is independent of k, $\Omega(\cdot)$, $W(\cdot)$, and the coefficients of $P_j(\cdot)$.

For $y=(y_1,\dots,y_n)\in\mathbf{R}^n$ we let $\tilde{y}=(y_1,\dots,y_{n-1})\in\mathbf{R}^{n-1}$. We shall consider the functions $F:\mathbf{R}\times\mathbf{R}^n\to\mathbf{R}$ of the form

$$F(t,y)=t^l q(\tilde{y})+W_1(t,y)+W_2(t) \tag{4.3}$$

where $q:\mathbf{R}^{n-1}\to\mathbf{R}$ is a polynomial, W_1 satisfies

$$\frac{\partial^l W_1}{\partial t^l}(t,y)\equiv 0, \tag{4.4}$$

and $W_2(\cdot)$ is an arbitrary function.

Lemma 4.6 ([9, Proposition 5.3])**.** *Let $\rho\in(0,1/4)$, $l\in\mathbf{N}$, $m\geq 0$, $q(\tilde{y})=\sum_{j=0}^m q_j(\tilde{y})$, where $q_j(\cdot)$ is a homogeneous polynomial of degree j on $\mathbf{R}^{n-1}$ for $0\leq j\leq m$. Let $F(t,y)$ be given by* (4.3) *and* (4.4)*. Suppose that $b\in L^\infty$ and $\Omega(\cdot)$ is a function satisfying*

$$\operatorname{supp}(\Omega)\subset\{w\in\mathbf{S}^{n-1};\ |w-\mathbf{e}|<\rho\} \tag{4.5}$$

and

$$\|\Omega\|_\infty\leq\rho^{-n+1}. \tag{4.6}$$

If we assume $q_m(\tilde{y})=\sum_{|\beta|=m}a_\beta\tilde{y}^\beta$ and $\|q_m\|=\sum_{|\beta|=m}|a_\beta|$, then there exists a positive constant C such that

$$\int_{2^k}^{2^{k+1}}\Big|\int_{\mathbf{S}^{n-1}}e^{iF(t,y)}\Omega(y)d\sigma(y)\Big|\frac{|b(t)|}{t}dt\leq C(2^{kl}\rho^m\|q_m\|)^{-\frac{1}{4ml}}. \tag{4.7}$$

The constant C may depend on l, m, n, and $b(\cdot)$, but it is independent of k, ρ, $W_1(\cdot,\cdot)$, $W_2(\cdot)$, and the coefficients of $q(\cdot)$.

Theorem 4.7. *Let $w=(x,z)\in\mathbf{R}^2$ and $\psi(w)$ be real-analytic on B_2^1 with $\psi(0)=\bigtriangledown\psi(0)=0$. Let $\mathbf{\Psi}(w)=(w,\psi(w))$ and $\mu^*_{\mathbf{\Psi}}$ be as defined in* (2.3)*. If $b\in L^\infty(\mathbf{R})$ and Ω is a regular atom defined in Definition* 4.1*, then*

$$\|\mu^*_{\mathbf{\Psi}}f\|_p\leq C_p\|f\|_p,\qquad 1<p\leq\infty \tag{4.8}$$

for $f\in L^p(\mathbf{R}^d)$, where C_p is independent of ζ and ρ.

Proof. For $p=\infty$, (4.8) is trivial; so we only consider $1<p<\infty$. We assume $\rho<R_0\leq\frac{1}{16}$, where R_0 is a small positive number to be fixed later. For $\rho\geq R_0$, the proof is similar (indeed easier) as one can see from the proof of Theorem 1.1 in [9]. We divide our discussion into two cases. □

Case 1: ψ is radial.

Since ψ is rotation-invariant, after a suitable rotation in $\mathbf{R}^2$, we may assume that $\operatorname{supp}(\Omega)\subset\{w\in\mathbf{S}^1;\ |w-\mathbf{e}|<\rho\}$. Let $\xi=(\xi_1,\xi_2,\xi_3)\in\mathbf{R}^3$. In Lemma 4.5, we take $l=1$ to get

$$|\hat{\mu}_{k,\mathbf{\Psi}}(\xi)|\leq C[2^k(|\xi_1|+|\xi_2|)\rho^4]^{-\frac{1}{8}} \tag{4.9}$$

for $k\in\mathbf{Z}_-$, where we used the fact that $\|\Omega\|_2=C\rho^{-\frac{1}{2}}$.

For $t\in[0,1]$, $|x|<\frac{2}{3}$, we write w in polar coordinates, namely, let $w=(tx,\ t\sqrt{1-x^2})$ so that locally we have $\mathbf{\Psi}=\{(tx,t\sqrt{1-x^2},\psi(tx,t\sqrt{1-x^2})),\ t\in[0,1],\ |x|<\frac{2}{3}\}$. Since ψ is radial, we have $\psi(tx,t\sqrt{1-x^2})=\psi(t)$. Since the power series expansion of $\sqrt{1-x^2}$ is $1-\frac{1}{2}x^2-\frac{1}{8}x^4-\cdots$, we define $\mathbf{\Psi}_1=(tx,t(1-\frac{1}{2}x^2),\psi(t))$. It follows that

$$|\hat{\mu}_{k,\mathbf{\Psi}}(\xi)-\hat{\mu}_{k,\mathbf{\Psi}_1}(\xi)|\le C[2^k(|\xi_1|+|\xi_2|)\rho^4] \tag{4.10}$$

for $k\in\mathbf{Z}_-$.

From Lemma 2.1, we see that (4.9) and (4.10) reduce $\mathbf{\Psi}$ to $\mathbf{\Psi}_1$ since one can take $a_0=2$, $T_0\xi=(\rho^4\xi_1,\rho^4\xi_2,0)$. Invoking Lemma 4.6 with $l=1$ and $m=2$, we have

$$|\hat{\mu}_{k,\mathbf{\Psi}_1}(\xi)|\le C[2^k|\xi_2|\rho^2]^{-\frac{1}{8}}.$$

If we define $\mathbf{\Psi}_2=(tx,t,\psi(t))$, then we have

$$|\hat{\mu}_{k,\mathbf{\Psi}_1}(\xi)-\hat{\mu}_{k,\mathbf{\Psi}_2}(\xi)|\le C[2^k|\xi_2|\rho^2].$$

Thus Lemma 2.1 reduces $\mathbf{\Psi}_1$ to $\mathbf{\Psi}_2$. We apply Lemma 4.6 again with $l=1$ and $m=1$ to get

$$|\hat{\mu}_{k,\mathbf{\Psi}_2}(\xi)|\le C[2^k|\xi_1|\rho]^{-\frac{1}{4}}.$$

Then we define $\mathbf{\Psi}_3=(0,t,\psi(t))$ so that we have

$$|\hat{\mu}_{k,\mathbf{\Psi}_2}(\xi)-\hat{\mu}_{k,\mathbf{\Psi}_3}(\xi)|\le C[2^k|\xi_1|\rho].$$

It follows that we have reduced $\mathbf{\Psi}_2$ to $\mathbf{\Psi}_3$.

For fixed x_1, let $g(x_2,x_3)=f(x_1,x_2,x_3)$. We observe that $\mu^*_{\mathbf{\Psi}_3}f(x_1,x_2,x_3)\le C\int_{S^1}M_\psi(|g|)(x_2,x_3)|\Omega(y)|d\sigma(y)\le CM_\psi(|g|)(x_2,x_3)\|\Omega\|_1\le CM_\psi(|g|)(x_2,x_3)$.

Applying Lemma 4.4 for $M_\psi(|g|)(x_2,x_3)$ and $1<p<\infty$, we have

$$\begin{aligned}
\|\mu^*_{\mathbf{\Psi}_3}f\|_p^p &= \int_{R^2}\int_{R^1}(\mu^*_{\mathbf{\Psi}_3}f)^p(x_1,x_2,x_3)dx_2dx_3dx_1\\
&\le C\int_{R^2}\int_{R^1}(M_\psi(|g|)(x_2,x_3))^pdx_2dx_3dx_1\\
&\le C\int_{R^2}\int_{R^1}|g(x_2,x_3)|^pdx_2dx_3dx_1\\
&\le C\int_{R^2}\int_{R^1}|f(x_1,x_2,x_3)|^pdx_1dx_2dx_3\\
&= C\|f\|_p^p.
\end{aligned}$$

This finishes the proof of case 1.

Case 2: ψ is not radial.

Let ζ be as in (2.1). Without loss of generality, we may assume that $|\zeta-\mathbf{e}|<\frac{1}{16}$. Let $\zeta=(y,\sqrt{1-y^2})$ so that $|y|<\frac{1}{16}$. It follows that if we let $\alpha(x)=\Omega(x,\sqrt{1-x^2})(1-x^2)^{-\frac{1}{2}}$, then $\operatorname{supp}(\alpha)\subset\{x\mid\ |x-y|<\rho\}$ and $\|\alpha\|\le 2\rho^{-1}$.

Let B_2^1 be the unit ball of $\mathbf{R}^2$. Since $\psi(w)$ is real-analytic on B_2^1, we have

$$\psi(w)=\sum_{j=2}^{\infty}\sum_{|\alpha|=j}c_\alpha w^\alpha. \tag{4.11}$$

Let $f(t,x)=\psi(tx,t\sqrt{1-x^2})$. It follows that

$$f(t,x)=\sum_{j=2}^{\infty}t^j[\sum_{n=0}^{\infty}a_{j,n}x^n], \tag{4.12}$$

where $a_{j,n}$ satisfies (3.7).

One can rewrite $f(t,x)$ as

$$f(t,x)=\sum_{j=2}^{\infty}t^j[\sum_{m=0}^{\infty}b_{j,m}(y)(x-y)^m], \tag{4.13}$$

where $b_{j,m}(y)$ is defined by (3.8).

We point out that the rotation $\tilde{r}_0$ on R^2 in Theorem 3.6 is equivalent to the translation of the variable x by y. Let $g(t,x,\xi',y)=t[\xi_1'x+\xi_2'\sqrt{1-x^2}]+\xi_3'f(t,x)$. We have

$$\hat{\mu}_{k,\mathbf{\Psi}}(\xi)=\int_{2^k}^{2^{k+1}}\int_{-\rho}^{\rho}e^{i|\xi|g(t,x,\xi',y)}\alpha(x)dx\frac{b(t)}{t}dt.$$

Since ψ is not radial, we see that $f(t,x)$ depends on x. Thus for each y with $|y|<\frac{1}{4}$, one can find $b_{j,m}(y)\neq 0$ for some $j\geq 2$ and some $m>0$. It follows that the function g satisfies the conditions (3.3) and (3.4). Now we fix R_0 so small that Theorem 3.6 can be applied. Thus, one can find $L,M\geq 1,\ N>0$ and $\alpha_0>0$ such that

$$|\hat{\mu}_{k,\mathbf{\Psi}}(\xi)|\leq C(|\xi|2^{kL}\rho^{M(L+1)})^{-\alpha_0} \tag{4.14}$$

if $k\in\mathbf{Z}^-,\ |k|>N,\ \ 0<\rho\leq R_0$.

Let

$$f_1(t,x)=\sum_{j=2}^{L-1}t^j[\sum_{m=0}^{\infty}b_{j,m}(y)(x-y)^m]+\sum_{L}^{\infty}t^j[\sum_{m=0}^{M(L+1)-1}b_{j,m}(y)(x-y)^m]$$

and define the mapping $\mathbf{\Psi}_1$ by

$$\mathbf{\Psi}_1(t,x)=(tx,t\sqrt{1-x^2},f_1(t,x)). \tag{4.15}$$

The above definition together with (4.13) yields

$$|\hat{\mu}_{k,\mathbf{\Psi}}(\xi)-\hat{\mu}_{k,\mathbf{\Psi}_1}(\xi)|\leq C(|\xi|2^{kL}\rho^{M(L+1)}) \tag{4.16}$$

if $0<\rho\leq R_0$, where C is independent of $k,\ \rho,\ \xi$ and y.

Having obtained (4.14) and (4.16), we have reduced $\mathbf{\Psi}$ to $\mathbf{\Psi}_1$ by choosing $T_0\xi=\rho^{M(L+1)}\xi$ and $a=2^L$. If the second summation of $f_1(t,x)$ is zero, (4.11), (4.12) and (4.13) imply that $\Psi_1=(w,\sum_{j=2}^{L-1}\sum_{|\alpha|=j}c_\alpha w^\alpha)$, which is covered by Theorem 7.4 in [9] since in this case $\mathbf{\Psi}_1$ is a polynomial mapping. Thus, we may assume that the second summation of $f_1(t,x)$ is not identically zero.

To reduce $\mathbf{\Psi}_1$ further, one can employ Theorem 3.7 to get the desired negative power estimate for $\hat{\mu}_{k,\mathbf{\Psi}_1}(\xi)$. If $\sum_L^\infty\sum_{m=0}^{M(L+1)-1}|b_{j,m}(y_0)|=0$ for some y_0, then one can find $d\in\mathbf{N}$ such that $b_{j,m}(y)=(y-y_0)^dc_{j,m}(y)$ and $\sum_L^\infty t^j[\sum_{m=0}^{M(L+1)-1}|c_{j,m}(y)|]>0$ in a neighborhood of y_0. Since $b_{j,m}(y)$ is analytic, by decomposing the interval of y into several subintervals if necessary, we may assume that y_0 is the only common zero of $b_{j,m}(y)$. Thus by Theorem 3.7, one can find $L_1\geq L,\ N>0$ and $\alpha_1>0$ such that

$$|\hat{\mu}_{k,\mathbf{\Psi}_1}(\xi)|\leq C(|\xi_3(y-y_0)^d|2^{kL_1}\rho^{M(L+1)-1})^{-\alpha_1} \tag{4.17}$$

if $k\in\mathbf{Z}_-,\ |k|>N,\ \ 0<\rho\leq\frac{1}{16}$.

Let

$$\begin{aligned} f_2(t,x) &= \sum_{j=2}^{L-1} t^j [\sum_{m=0}^{\infty} b_{j,m}(y)(x-y)^m] + \sum_{j=L}^{L_1-1} t^j [\sum_{m=0}^{M(L+1)-1} b_{j,m}(y)(x-y)^m] \\ &+ \sum_{L_1}^{\infty} t^j [\sum_{m=0}^{M(L+1)-2} b_{j,m}(y)(x-y)^m], \end{aligned}$$

where the $\sum_{j=L}^{L_1-1}$ term is zero if $L_1 = L$. We define the mapping $\mathbf{\Psi}_2$ by

$$\mathbf{\Psi}_2 = (tx, t\sqrt{1-x^2}, f_2(t,x)) \tag{4.18}$$

It follows from (4.15) and (4.18) that

$$|\hat{\mu}_{k,\mathbf{\Psi}_1}(\xi) - \hat{\mu}_{k,\mathbf{\Psi}_2}(\xi)| \le C(|\xi_3(y-y_0)^d|2^{kL_1}\rho^{M(L+1)-1}) \tag{4.19}$$

if $0 < \rho \le R$, where C is independent of $k,\ \rho,\ \xi$ and y.

Then one applies Lemma 2.1 with $T_1\xi = (y-y_0)^d\rho^{M(L+1)-1}\xi_3$ and $a_1 = 2^{L_1}$ to see that $\mathbf{\Psi}_1$ has been reduced to $\mathbf{\Psi}_2$. Repeating this process at most $M(L+1)$ times, one can reduce $\mathbf{\Psi}$ to a mapping $\mathbf{\Phi}$ given by

$$\mathbf{\Phi} = (tx, t\sqrt{1-x^2}, g(t,x,y)) \tag{4.20}$$

with

$$\begin{aligned} g(t,x,y) &= \sum_{j=2}^{L-1} t^j [\sum_{m=0}^{\infty} b_{j,m}(y)(x-y)^m] + \sum_{L}^{L_1-1} t^j [\sum_{m=0}^{M(L+1)-1} b_{j,m}(y)(x-y)^m] \\ &+ \sum_{l=1}^{M(L+1)-1} \sum_{L_l}^{L_{l+1}-1)} t^j [\sum_{m=0}^{M(L+1)-(l+1)} b_{j,m}(y)(x-y)^m] + \sum_{j=L_{M(L+1)}}^{\infty} b_{j,0} t^j, \end{aligned}$$

where $L_1 \le L_2 \le \cdots \le L_{M(L+1)}$.

Based on Lemma 4.6 and Lemma 2.1, one then applies the argument that reduced $\mathbf{\Psi}_1$ to $\mathbf{\Psi}_2$ finitely many times to see that $\mathbf{\Phi}$ can be reduced to $\mathbf{\Phi}_1$, where $\mathbf{\Phi}_1$ is defined by

$$\mathbf{\Phi}_1 = (tx, t\sqrt{1-x^2}, g_1(t,x,y)) \tag{4.21}$$

with

$$g_1(t,x,y) = \sum_{j=2}^{L-1} t^j [\sum_{m=0}^{\infty} b_{j,m}(y)(x-y)^m] + \sum_{j=L}^{\infty} b_{j,0}(y) t^j.$$

Again from (4.11), (4.12) and (4.13), we see that $\mathbf{\Phi}_1 = (w,\ \sum_{j=2}^{L-1}\sum_{|\alpha|=j} c_\alpha w^\alpha + h_1(|w|))$, where $h(|w|) = \sum_{j=L}^{\infty} b_{j,0}(y)|w|^j$. Similar to the process of reducing $\mathbf{\Psi}$ to $\mathbf{\Psi}_1$ and then to $\mathbf{\Psi}_2$, we can combine Lemma 4.2, Lemma 4.3 and Lemma 2.2 to reduce $\mathbf{\Phi}_1$ to $\mathbf{\Phi}_2$ and then to $\mathbf{\Phi}_3$, where

$$\mathbf{\Phi}_2(w) = (w, h_2(|w|)) \tag{4.22}$$

with $h_2(|w|) = \sum_{j=2}^{\infty} b_{j,0}(y)|w|^j$ and

$$\mathbf{\Phi}_3(w) = (0, |w|, h_2(|w|)) = (0, t, h_2(t)). \tag{4.23}$$

We point out that one cannot apply Lemma 4.4 directly to $\mathbf{\Phi}_3$ since the coefficients of $h_2(t)$ are dependent on y and our desired estimate (4.8) should be independent of y. Thus we need to apply Lemma 2.1 to reduce $\mathbf{\Phi}_3$ further. From the definition of $\mu^*_{\mathbf{\Phi}_3}$, we see that, without loss of generality, we may assume that $b(t) \equiv 1$. It follows that

$$\hat{\mu}_{k,\mathbf{\Phi}_3}(\xi) \quad = \quad \int_{\mathbf{S}^1} [\int_{2^k}^{2^{k+1}} e^{i(\xi_2 t + \xi_3 h_2(t))} \frac{1}{t} dt] |\Omega(w)| d\sigma(w).$$

Also, if necessary we can pull out a factor $(y - y_0)$ from $b_{j,0}(y)$ so that we may assume $\sum_{j=2}^{\infty} |b_{j,0}(y)| > 0$ for all $|y| \leq \frac{1}{16}$. This combined with van der Corput's lemma yields that there exist $J,\ J_1,\ N \in \mathbf{N} \setminus \{1\}$ such that

$$\begin{aligned} |\int_{2^k}^{2^{k+1}} e^{i(\xi_2 t + \xi_3 h_2(t))} \frac{1}{t} dt| &= |\int_1^2 e^{i(\xi_2 2^k t + \xi_3 h_2(2^k t))} \frac{1}{t} dt| \\ &\leq C(|\xi_3| 2^{Jk})^{-\frac{1}{J_1}} \end{aligned} \tag{4.24}$$

for all $k \in \mathbf{Z}_-,\ |k| > N$.

Since $\Omega \in L^1(\mathbf{S}^1)$, (4.24) yields

$$|\hat{\mu}_{k,\mathbf{\Phi}_3}(\xi)| \leq C(|\xi_3| 2^{Jk})^{-\frac{1}{J_1}} \tag{4.25}$$

for all $k \in \mathbf{Z}_-,\ |k| > N$, where C is independent of $k,\ \xi,\ \rho$ and y.

We define $\mathbf{\Phi}_4$ by

$$\mathbf{\Phi}_4(w) = (0, |w|, \sum_{j=2}^{J} b_{j,0}(y) |w|^j). \tag{4.26}$$

It follows from (4.23) and (4.26) that

$$|\hat{\mu}_{k,\mathbf{\Phi}_3}(\xi) - \hat{\mu}_{k,\mathbf{\Phi}_4}(\xi)| \leq C(|\xi_3| 2^{Jk}) \tag{4.27}$$

for all $k \in \mathbf{Z}_-,\ |k| > N$, where C is independent of $k,\ \rho,\ \xi$ and y.

From (4.25) and (4.27) we see that $\mathbf{\Phi}_3$ has been reduced to $\mathbf{\Phi}_4$, whose L_p estimate can be derived easily from a result in [23], pages 476–478, where the L^p estimate is independent of the coefficients of the polynomials of t.

This ends the proof of Theorem 4.7.

Proof of Theorem 1.2. From Lemma 4.2, we may assume that Ω is a regular atom. For $k \in \mathbf{Z}_-$, let $\sigma_{k,\mathbf{\Psi}}$ and $\mu_{k,\mathbf{\Psi}}$ be defined by (2.1) and (2.2). Since U is bounded and the kernel $K \in L^1$ on $U \setminus B^1_2$, from the definition (1.3) of $T_{\mathbf{\Psi}}$, we may assume that $T_{\mathbf{\Psi}} f = \sum_{-\infty}^{0} \sigma_k * f$. In the proof of Theorem 4.7, after finitely many steps), we reduced the mapping $\mathbf{\Psi}$ to $\mathbf{\Psi}_3$, where in the radial case, $\mathbf{\Psi}_3 = (0, t, \psi(t))$ and in the nonradial case, $\mathbf{\Psi}_3 = (0, t, h_2(t))$. Since in both cases $\mathbf{\Psi}_3$ depends only on the radial variable t, we see that $\sigma_{k,\mathbf{\Psi}_3} = 0$ since $\int_{S^1} \Omega(w) dw = 0$. Without loss of generality, we may assume that we reduced $\mathbf{\Psi}$ to $\mathbf{\Psi}_3$ by three steps (in the radial case, exactly three steps). For later convenience we denote $\mathbf{\Psi}$ as $\mathbf{\Psi}_0$. Also, we remark that all the estimates obtained in the proof of Theorem 4.7 for μ_k are valid for σ_k. As indicated for the radial case (see (4.9) and (4.10)), for $i = 0,\ 1,\ 2$, one can find $a_i > 1,\ 0 < \alpha_i < 1$ and linear transformations $T_i :\ R^3 \to R^3$ such that

$$|\hat{\sigma}_{k,\mathbf{\Psi}_i}(\xi)| \leq C(a_i^k |T_i \xi|)^{-\alpha_i}, \tag{4.28}$$

$$|\hat{\mu}_{k,\mathbf{\Psi}_i}(\xi) - \hat{\mu}_{k,\mathbf{\Psi}_{i+1}}(\xi)| \leq C(a_i^k |T_i \xi|). \tag{4.29}$$

Let $s_i = \text{rank}(T_i)$. As in (2.8), there are nonsingular linear transformations $G_i: R^3 \to R^3$ be such that

$$|\pi^3_{s_i} G_i \xi| \le |T_i \xi| \le (3 - s_i + 1)|\pi^3_{s_i} G_i \xi| \tag{4.30}$$

for every $\xi \in \mathbf{R}^3$.

Let $S(\mathbf{R}^{s_i})$ be the Schwartz class functions in $\mathbf{R}^{s_i}$. We choose and fix a function $\varphi_i \in C_0^\infty(\mathbf{R}^{s_i})$ such that $\varphi_i(t) \equiv 1$ for $|t| \le 1/2$, and $\varphi_i(t) \equiv 0$ for $|t| \ge 1$. Let $\Phi_i \in \mathcal{S}(\mathbf{R}^{s_i})$ such that $\hat{\Phi}_i = \varphi_i$.

For $k \in \mathbf{Z}_-$ and $i = 0,\ 1,\ 2$, we define the measures $\tau_{k,i}$ on $\mathbf{R}^3$ by

(4.31)
$$\hat{\tau}_{k,0}(\xi) = \hat{\sigma}_{k,\mathbf{\Psi}_0}(\xi) - \phi_0(a_0^k \pi^3_{s_0} G_0 \xi) \hat{\sigma}_{k,\mathbf{\Psi}_1}(\xi),$$

(4.32)
$$\hat{\tau}_{k,1}(\xi) = \phi_0(a_0^k \pi^3_{s_0} G_0 \xi) \hat{\sigma}_{k,\mathbf{\Psi}_1}(\xi) - \phi_0(a_0^k \pi^3_{s_0} G_0 \xi) \phi_1(a_1^k \pi^3_{s_1} G_1 \xi) \hat{\sigma}_{k,\mathbf{\Psi}_2}(\xi),$$

(4.33)
$$\begin{aligned}\hat{\tau}_{k,2}(\xi) &= \phi_0(a_0^k \pi^3_{s_0} G_0 \xi) \phi_1(a_1^k \pi^3_{s_1} G_1 \xi) \phi_2 \hat{\sigma}_{k,\mathbf{\Psi}_2}(\xi) \\ &\quad - \phi_0(a_0^k \pi^3_{s_0} G_0 \xi) \phi_1(a_1^k \pi^3_{s_1} G_1 \xi) \phi_2(a_2^k \pi^3_{s_2} G_2 \xi) \hat{\sigma}_{k,\mathbf{\Psi}_3}(\xi).\end{aligned}$$

From (4.28) to (4.33) and the choice of $\phi_i(t)$, for $i = 0,\ 1,\ 2$, we have

$$|\hat{\tau}_{k,i}(\xi)| \le C[\min\{a_i^k |T_i \xi|,\ (a_i^k |T_i \xi|)^{-1}\}]^{\alpha_i}. \tag{4.34}$$

The proof of Theorem 4.7 yields that for each i we have

$$\|\sigma^*_{\mathbf{\Psi}_i} f\|_p \le C_p \|f\|_p, \qquad 1 < p < \infty. \tag{4.35}$$

Repeating the argument contained in the proof of Lemma 2.1 ((2.12) –(2.19)) $(i+1)$ times for τ_i^*, we see that

$$\|\tau_i^* f\|_p \le C_p \|f\|_p, \qquad 1 < p < \infty. \tag{4.36}$$

It follows from (4.34) and (4.36) that one can apply Lemma 4.3 for $\tau_{k,i}$ to get

$$\|\sum_{-\infty}^{0} \tau_{k,i} * f\|_p \le C_p \|f\|_p, \qquad 1 < p < \infty. \tag{4.37}$$

Now the inequality (1.8) in Theorem 1.2 is derived from (4.37) since $\sigma_{k,\mathbf{\Psi}} = \tau_{k,0} + \tau_{k,1} + \tau_{k,2}$. Here we used the fact that $\sigma_{k,\mathbf{\Psi}_3} = 0$ for all $k \in \mathbf{Z}_-$.

This is the end of the proof of Theorem 1.2. □

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Department of Mathematical Sciences, University of Wisconsin-Milwaukee, Milwaukee, Wisconsin 53201

E-mail address: fan@csd4.csd.uwm.edu

Department of Mathematics, Southwest Missouri State University, Springfield, Missouri 65804

E-mail address: kag026f@smsu.edu

Department of Mathematics, University of Pittsburgh, Pittsburgh, Pennsylvania 15260

E-mail address: yibiao+@pitt.edu